

Studies of bosons in optical lattices in a harmonic potential

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Abstract. We present a theoretical study of Bose condensation and specific heat of non-interacting bosons in finite lattices in harmonic potentials in one, two, and three dimensions. We numerically diagonalize the Hamiltonian to obtain the energy levels of the systems. Using the energy levels thus obtained, we investigate the temperature dependence, dimensionality effects, lattice size dependence, and evolution to the bulk limit of the condensate fraction and the specific heat. Some preliminary results on the specific heat of fermions in optical lattices are also presented. The results obtained are contextualized within the current experimental and theoretical scenario.

PACS. 03.75.Lm Tunneling, Josephson effect, Bose-Einstein condensates in periodic potentials, solitons, vortices, and topological excitations – 03.75.Nt Other Bose-Einstein condensation phenomena – 03.75.Hh Static properties of condensates; thermodynamical, statistical, and structural properties

1 Introduction

Bosons in optical lattices provides a microscopic laboratory for exploration of the properties of many-boson systems with unprecedented control on number of bosons per site, boson kinetic energy, and boson-boson interaction strength [1–5]. Experimentalists have explored [1–4] quantum phase transitions, excitation spectra, condensate fraction as a function of boson-boson interaction strength, and properties of boson Tonks gas in one, two, and three dimensional optical lattices. Theoretical studies [5–18] of bosons in optical lattices thus far mainly concentrated on quantum phase transitions between various possible phases such as Mott insulator phase, density-wave phase, and Bose-condensed phase. Finite temperature properties of bosons in the combined optical lattice potential and the confining harmonic potential have not received much attention except in some studies [19,20] where, again, the focus is on Mott transition.

In this paper, we present a theoretical study of bosons in optical lattices in a harmonic potential in one, two, and three dimensional optical lattices. In particular, we study the temperature dependence of the condensate fraction and specific heat. This study is useful for the following reasons. After the discovery [21–23] of Bose-Einstein condensation in harmonic traps (for reviews see Refs. [24–27]), one of the important studies done was to measure the

temperature dependence of the condensate fraction and its comparison with theory, and such measurements for bosons in optical lattices can be expected in near future. Thermodynamic properties of bosons in optical lattices are of considerable interest in their own right. Though no measurement of specific heat of bosons in a harmonic potential and of bosons in a lattice with harmonic potential have been reported thus far, such measurements have already been reported [28] for fermions in harmonic potential, and it is not unreasonable to expect such measurements for bosons as well. Furthermore, experiments are performed on bosons in *finite* lattices, and so it would be useful to understand the properties of bosons in finite lattices to see finite size effects and their evolution towards the bulk limit. The motivations and relevance of the studies presented in this paper are, hopefully, clear to the reader from the foregoing.

As stated, we present studies of lattice bosons in harmonic traps. We have considered non-interacting bosons in a periodic lattice in one-dimension (1d), a square lattice in two-dimensions (2d), and a cubic lattice in three-dimensions (3d). The bosons are considered to be moving in the combined lattice and harmonic potentials. We numerically diagonalize the Hamiltonian matrix for bosons in finite lattices in a harmonic potential, and concentrate on calculations of the temperature dependence of ground state occupancy, specific-heat, finite-size effects, and dimensionality effects. This work is presented in Sections 2

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and 3. In Section 4, some preliminary results on our recent work on fermions in optical lattices in a harmonic potential is presented, and conclusions are given in Section 5.

2 Model and method

In this Section, we will describe the model and the method followed. Consider non-interacting bosons under the combined influence of a periodic optical lattice potential and an overall harmonic confining potential. The Hamiltonian of this system is,

$$H = -t \sum_{\langle ij \rangle} (c_i^\dagger c_j + c_j^\dagger c_i) + K \sum_i r_i^2 n_i - \mu \sum_i n_i, \quad (1)$$

where t is the kinetic energy gain when a boson hop from site i to its nearest neighbor site j in the optical lattice, c_i^\dagger is the boson creation operator, K (>0) the strength of the harmonic confining potential, \mathbf{r}_i the locator of the site i with respect to the origin which is at the center of the lattice, $n_i = c_i^\dagger c_i$ the boson number operator, and μ the chemical potential. For bosons under the influence of only the harmonic potential Bose condensation is by now well understood [24–27]. Our aim is to consider the case of finite t and finite K . After writing the Hamiltonian (Eq. (1)) in the single particle site basis, we numerically diagonalize it for various values of K and for various lattice sizes in 1d, 2d, and 3d. The energy levels thus obtained are used in calculations of the ground state occupancy and specific heat as a function of temperature. We will see that any non-zero value of K leads to substantial changes in both these properties. We have calculated the energy levels and the bosonic properties using open boundary conditions. It may be noted here that for the 1d case, the eigenfunctions and eigenenergies may be exactly obtained in terms of the Mathieu functions as shown by Rey and collaborators [29]. They have given approximate analytical expressions for the eigenenergies and eigenfunctions. We compared our eigenenergies for 1d systems with their approximate result (Eq. (15) in their paper) in the tunneling dominated regime and found that their eigenenergies are very close to our exact results for large values of t/k .

The ground state occupancy and the specific heat are calculated in the following way. The population $N(E_i)$ of a state with energy E_i is given by the Bose distribution

$$N(E_i) = \frac{1}{e^{\beta(E_i - \mu)} - 1}, \quad (2)$$

where $\beta = 1/k_B T$ and k_B is the Boltzmann constant. The chemical potential is determined from the number equation

$$N = \sum_{i=0}^{i_m} N(E_i), \quad (3)$$

where E_i with $i = 0, 1, 2, \dots, i_m$ denotes the energy levels from the lowest to the highest level of a boson in a finite lattice in the presence of the harmonic confining potential.

For a given number of bosons, at any temperature, the chemical potential and the population in the lowest level (N_0) are determined using equations (2) and (3).

The total energy of the system of bosons is given by

$$E_{tot} = \sum_{i=0}^{i_m} N(E_i) E_i. \quad (4)$$

The temperature derivative of E_{tot} gives the specific heat

$$C_v = \frac{1}{k_B T^2} \left[I_3 - \frac{I_2 I_1}{I_0} \right], \quad (5)$$

where,

$$I_0 = \sum_{i=0}^{i_m} F(E_i - \mu), \quad (6)$$

$$I_1 = \sum_{i=0}^{i_m} E_i F(E_i - \mu), \quad (7)$$

$$I_2 = \sum_{i=0}^{i_m} (E_i - \mu) F(E_i - \mu), \quad (8)$$

$$I_3 = \sum_{i=0}^{i_m} E_i (E_i - \mu) F(E_i - \mu). \quad (9)$$

In the above equations

$$F(E_i - \mu) = \frac{e^{\beta(E_i - \mu)}}{[e^{\beta(E_i - \mu)} - 1]^2}. \quad (10)$$

3 Results and discussions on bosons

Before presenting the results of our calculations, some general remarks are in order. In the absence of a harmonic confining potential, the bosons in a periodic lattice undergo a Bose condensation phase transition only in 3d where the boson Density Of States (DOS) in the thermodynamic limit vanishes at the bottom of the band. The phase transition is not possible for bosons in 1d or 2d periodic lattices since the DOS at the bottom of the band either diverges or remains finite, and the condition $N_0 = 0$ and $\mu = E_0$ is not satisfied at any temperature. Here, E_0 is the energy of the lowest level or bottom of a band. For free bosons in a harmonic confining potential, the phase transition is possible in 2d but not in 1d [30]. For finite size systems and for finite number of particles there is no true phase transition using the above criteria. However, at sufficiently low temperatures, a macroscopic number of particles will occupy the ground state and thus Bose condensation will occur [31]. With these general remarks, we go to our results.

3.1 One dimensional case

In Figure 1, the variation of the condensate fraction (N_0/N) with temperature is shown for 1000 bosons in a

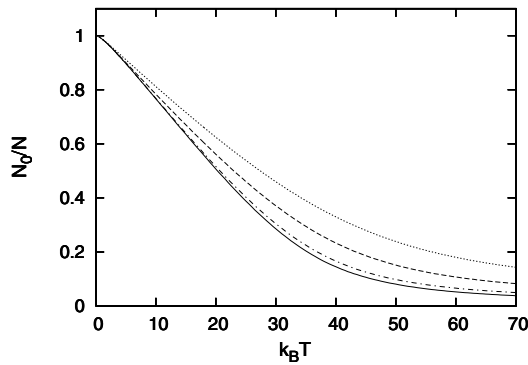


Fig. 1. The variation of condensate fraction with temperature for 1000 bosons in 1d finite lattices in a harmonic potential with $k = 0.01$. The size of the lattices are: $N_l = 30$ (dots), 50 (dashes), 100 (dash-dot), and 1000 (solid line). $k_B T$ is expressed in energy units of $t = 1$.

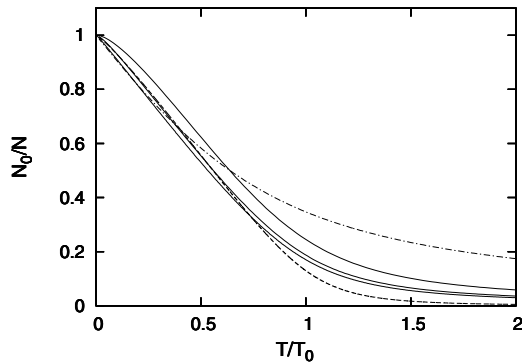


Fig. 2. The variation of condensate fraction with temperature. Solid lines are for bosons in a 1d finite lattice of size 1000 in a harmonic potential with $k = 0.01$ for boson numbers $N = 100$ (top), 1000 (middle), 10000 (bottom). The dashed line is for 10000 free bosons in a harmonic trap with $k = 0.01$. The dash-dot line is for 10000 bosons in a 1d lattice of 1000 sites. In this and other figures T_0 is the condensation temperature (see text).

1d lattice (of lattice constant a) for $k = Ka^2 = 0.01$ (in the energy scale of $t = 1$) for different lattice sizes (N_l). The occupancy in the lowest level is considerable even at high temperatures for smaller size systems and it decreases with increasing size. For $k = 0.01$, no size effect is seen for $N_l \geq 400$. We have presented a curve for $N_l = 1000$ which may be considered as an infinite size result for this value of k .

In Figure 2 we have shown the temperature dependence of the condensate fraction (N_0/N) for a one dimensional lattice of size $N_l = 1000$ in presence of harmonic potential of strength $k = 0.01$ for different numbers of bosons. The N_0/N is plotted against T/T_0 . Here, T_0 is determined by setting $N_0 = 0$ and $\mu = E_0$ in the number equation (Eq. (3)) to obtain

$$N = \sum_{i=1}^{i_m} \frac{1}{e^{(E_i - E_0)/k_B T_0} - 1}. \quad (11)$$

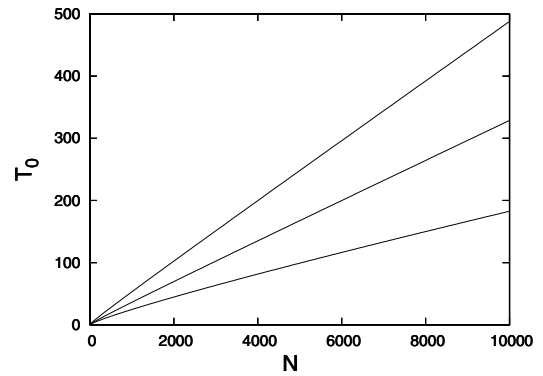


Fig. 3. The variation of T_0 (in energy units of $t = 1$) with N for bosons in a 1d lattice of size 1000 in a 1d harmonic trap with $k = 0.02$ (top), 0.01 (middle), and for free bosons in a harmonic trap for $k = 0.01$ (bottom).

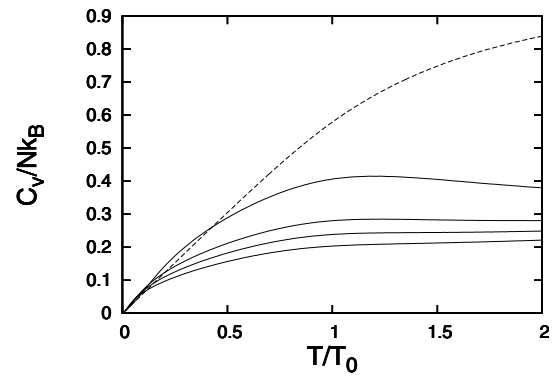


Fig. 4. Temperature dependence of the specific heat per particle for 600 bosons. The dashed line is for free bosons in a harmonic potential. Solid lines are for bosons in a 1d optical lattice of size 1000 in a harmonic potential for (from top to bottom): $k = 0.001$ (top), 0.005, 0.01, and 0.02 (bottom).

Note that in the thermodynamic limit and when the DOS vanishes at the bottom of the boson energy band, T_0 gives the phase transition temperature. For finite size lattices T_0 may be considered as the *condensation temperature*. In the absence of a harmonic trap, N_0/N is appreciable for bosons in a 1d lattice even at $T = 2T_0$ (dash-dot curve in Fig. 2). It decreases rapidly with the application of the harmonic potential and also decreases with increasing number of bosons for a fixed value of k . For a Bose gas in a harmonic trap (dashed curve in Fig. 2), N_0/N almost vanishes at high temperatures. In Figure 3, the variation of T_0 with number of bosons (N) is shown for different values of the harmonic potential. The T_0 increases almost linearly with N . For a fixed N , it increases with increasing strength of the harmonic potential.

In Figure 4, we have plotted the specific heat per particle against T/T_0 for different values of k for $N_l = 1000$ and $N = 600$. It is seen that the specific heat decreases with increasing value of k . For $k = 0.001$, a broad peak is seen in the specific heat, which is a signature of the finite size of the lattice. For such a small value of k , even a size of 1000 sites is not enough to confine the bosons within the

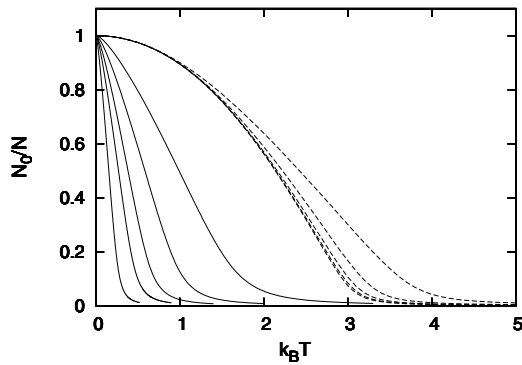


Fig. 5. Size effect in 2d for $N = 540$. The dashed curves are for $k = 0.01$ and lattice sizes: 30×30 (top), 40×40 , 50×50 , 60×60 , and 80×80 (bottom). Solid curves are for lattice bosons without harmonic potential for lattice sizes: 30×30 (top), 40×40 , 50×50 , 60×60 , and 80×80 (bottom). $k_B T$ is measured in energy units of $t = 1$.

harmonic trap (i.e., some of the bosons may occupy the boundary sites), hence finite size effects would show up. As the strength of the harmonic potential increases, the number bosons at high-energy sites at the edges of the lattice is insignificant and the finite size effect disappears. We have also shown C_v for bosons under the influence of only the harmonic potential. One notices at once that while the dependence of C_v on the scaled temperature (T/T_0) for free bosons in a harmonic potential is independent of k , the same is not true for lattice bosons. It is found that, switching on the lattice potential substantially reduces the C_v . We also note that reducing the strength of harmonic potential for the lattice bosons drives the system towards the limit of free bosons in a harmonic potential, which is counter intuitive to what one might naively expect. In the very low temperature range, all the curves are very close to each other. This can be understood if we look at the single particle spectrum of a large lattice. In this case, the low energy part of the band is approximately parabolic, and so one can transform the problem to a free particle in a harmonic potential. Hence one expects the properties of lattice bosons in a harmonic trap will be close to that of free bosons in harmonic trap at sufficiently low temperatures.

3.2 Two dimensional case

In this section we present our results on bosons in two dimensional optical lattices in a 2d harmonic potential. These results are presented in Figures 5–10. Figure 5 shows the variation of the condensate fraction with temperature for different lattice sizes for $k = 0$ and 0.01 , respectively. A macroscopic condensation occurs at a lower temperature for larger size of the system. The dashed lines in Figure 5 exhibit the effect of harmonic potential on lattice bosons in 2d. In the presence of a harmonic confining potential, N_0/N versus T curves become almost size independent for lattice sizes of 80×80 and larger, unlike the

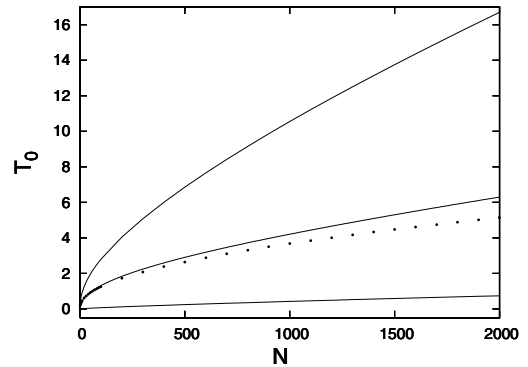


Fig. 6. The variation of T_0 (in energy units of $t = 1$) with number of bosons. Solid lines are for an 80×80 lattice and with $k = 0.05$ (top), 0.01 (middle), and $k = 0$ (bottom). The dots are for bosons with only the harmonic trap potential with $k = 0.01$.

case of $k = 0$ (solid lines) where the condensation temperature decreases rapidly with the size of the system. For largest lattice size shown, in presence of the harmonic potential our results may correspond to the results in the thermodynamic limit. It may be mentioned that for larger number of bosons a larger lattice size is needed to reach the infinite size limit. The T_0 is clearly seen to be considerably increased with the introduction of the harmonic potential. We also note that the condensate fraction grows faster with decreasing temperature in 2d as compared to that in 1d (see Figs. 1 and 2). In Figure 6, we have shown the variation of T_0 with boson number. The figure shows that, for a fixed lattice size and N , T_0 increases with increasing strength of the harmonic confinement potential. It also shows that the bosons under the combined effect of lattice and harmonic potentials have a higher transition temperature compared to bosons with either of the potentials absent. In comparison with 1d, T_0 is found to increase rapidly for small N and a monotonic increase for larger N , clearly showing a dimensionality effect.

In Figure 7, we have exhibited the effect of lattice size on the specific heat of bosons for a fixed N and k . For small lattice sizes, we find a flat region in C_v for temperatures below T_0 . With increasing lattice size the flat region evolves into a peak which becomes sharper as the size of the lattice approaches the bulk limit. If one compares the C_v results for 80×80 , 100×100 , 150×150 lattices, one finds that in the low temperature range the value of C_v is same for all the three curves, while in the high temperature range C_v is smaller for smaller size systems. The latter is a signature of the size effect. For higher temperatures more and more bosons would occupy the boundary sites of the lattice for a given value of k leading to appreciable finite size effects. Given the great control experimentalists have demonstrated, these effects may be within the reach of experimental observability in near future.

In order to get an understanding of the effect of harmonic potential strength on the specific heat of lattice bosons, we have shown in Figure 8, C_v as a function of T/T_0 for various values of k . This figure shows that the

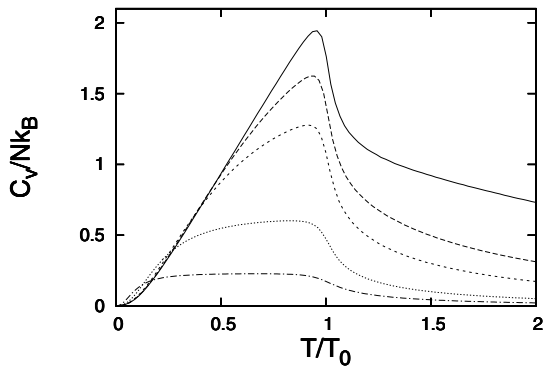


Fig. 7. Lattice size effect on specific heat in 2d for 3840 bosons in a harmonic trap with $k = 0.01$. The curves are for lattice sizes: 150×150 (solid line), 100×100 (long dashes line), 80×80 (short dashed line), 50×50 (dotted line), and 30×30 (dash-dot line).

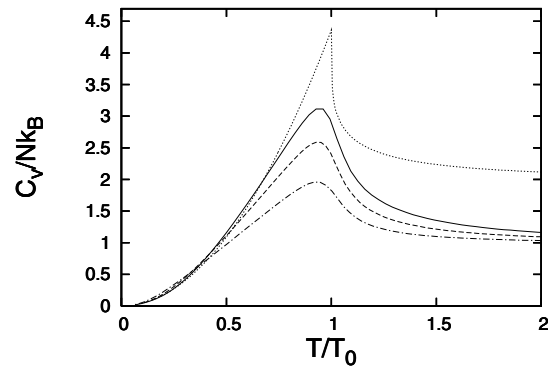


Fig. 9. Specific heat per boson in 2d for 540 bosons in a 150×150 lattice in a harmonic potential with $k = 0.01$ (solid line), $k = 0.02$ (dashed line), and $k = 0.05$ (dash-dot line). The dotted curve (top) is for bosons in 2d harmonic trap and is independent of k .

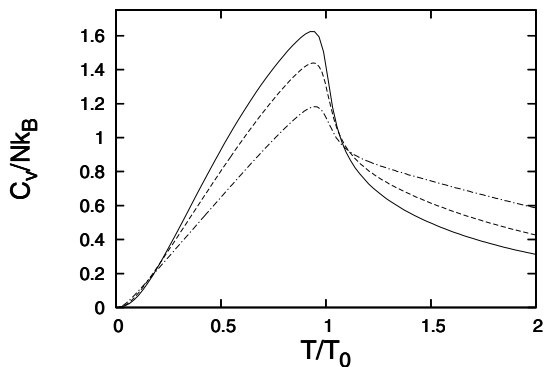


Fig. 8. Specific heat per boson in 2d for 3840 bosons in a 100×100 lattice in a harmonic potential with $k = 0.01$ (solid line), $k = 0.02$ (dash line), and $k = 0.05$ (dash-dot line).

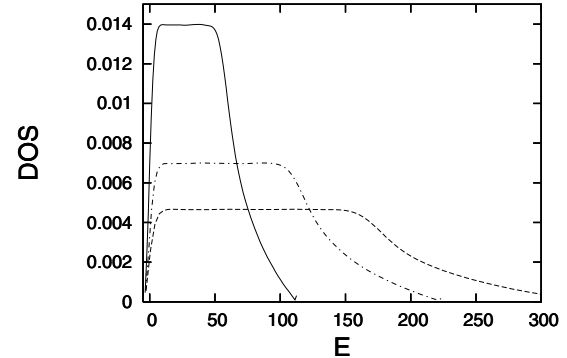


Fig. 10. Normalized DOS per site versus energy for an 100×100 lattice for $k = 0.01$ (solid line), 0.02 (dash-dot line), and 0.03 (dashed line).

C_v curves for different k values intercept at a temperature $T/T_0 \approx 1.1$ for the boson number ($N = 3840$) considered. Below this temperature C_v is lower for higher k , while above it increasing k increases C_v . This is in contrast to the behavior of bosons in 1d (Fig. 4) where increasing k decreases C_v except at very low temperatures. The peak in C_v curve is seen to decrease with increasing harmonic confinement. The behavior of C_v shown in Figure 8 results from an interplay of the effects of the harmonic potential and finite size of the lattice. The cross-over effect for $T/T_0 \geq 1.1$ occurs because in this temperature range bosons are delocalized over the entire lattice so that they sense the existence of boundaries. This is clear from Figure 9 in which C_v is shown for a larger lattice (150×150) and one notices that the crossing has disappeared in the temperature range shown.

In order to get an insight into the dimensionality effects, we calculated the one boson density of states for a lattice boson in a harmonic potential. The normalized DOS for the 2d case is shown in Figure 10. In the absence of the harmonic potential, the DOS has a van-Hove singularity for a 2d square lattice. Any finite harmonic potential leads to drastic changes in DOS by wiping out the

van-Hove singularity and flattening the DOS in an energy range which increases with increasing harmonic potential strength (k). This range also increases with increasing size of the lattice (not shown in the figure). The DOS at the bottom of the band vanishes as the size of the lattice approaches the infinite limit, and in the low energy range it shows almost linear increase with energy before taking the constant value. The DOS for bosons in a 2d lattice in presence of a harmonic trap may thus be approximately given by

$$\begin{aligned} \rho(E) &= \gamma E \quad \text{for } 0 \leq E \leq E_1 \\ &= \gamma E_1 \quad \text{for } E_1 \leq E \leq E_2, \end{aligned} \quad (12)$$

where all the energies are measured from the bottom of the energy spectrum. It is found that γ is strongly k dependent and decreases with increasing k , while E_1 is almost k independent. For free bosons in a 2d harmonic trap, the DOS has a linear dependence on E in the entire energy range ($0 \leq E \leq \infty$). For $\rho(E) \propto E^{\alpha-1}$, the specific heat per boson for $T \leq T_0$ is given by [27]

$$\frac{C_v}{Nk_B} = \alpha(\alpha + 1) \frac{\zeta(\alpha + 1)}{\zeta(\alpha)} \left(\frac{T}{T_0} \right)^\alpha, \quad (13)$$

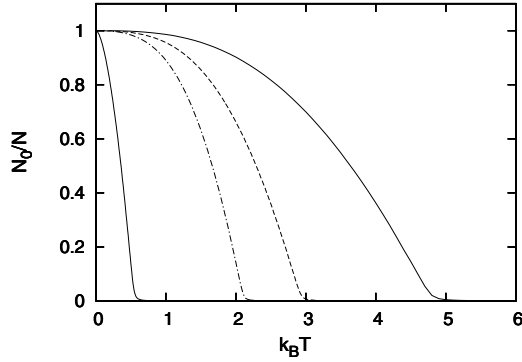


Fig. 11. Temperature dependence of condensate fraction for 2500 bosons in a cubic lattice with $50 \times 50 \times 50$ lattice sites and harmonic potential with $k = 0.05$ (top), $k = 0.02$ (dashed), $k = 0.01$ (dash-dot), and $k = 0$ (bottom).

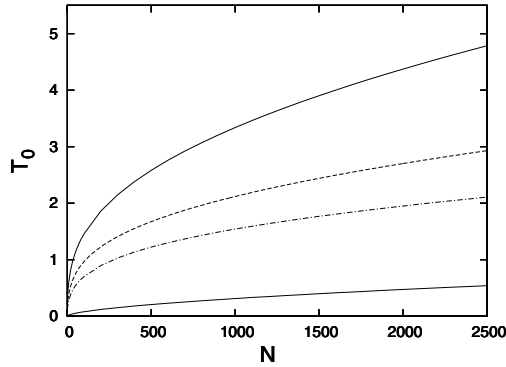


Fig. 12. Bosons number dependence of T_0 for bosons in a cubic $50 \times 50 \times 50$ lattice with $k = 0.05$ (top), $k = 0.02$ (dashed), $k = 0.01$ (dash-dot), and $k = 0$ (bottom).

where $\zeta(\alpha) = \sum_{n=1}^{\infty} (1/n^\alpha)$ is the Riemann zeta function. Since for free bosons in a harmonic trap $\alpha = 2$, C_v shows a $(T/T_0)^2$ behavior for $T \leq T_0$. This feature is seen in our calculated results (the dotted curve in Fig. 9). For the lattice bosons in a harmonic trap in the low temperature region, only the low energy linear part of the DOS is occupied. Hence C_v shows behavior similar to that of free bosons in a harmonic trap. When the temperature is appreciable so that the flat portion of the DOS is also occupied, the effective value of α decreases. This results in a slower increase of C_v with T/T_0 and lower value of C_v compared to the free bosons case. These features are prominent in Figure 9 for higher values of k which have higher T_0 values.

3.3 Three dimensional case

In Figures 11–13, we have presented our results for bosons in 3d cubic lattices in a harmonic potential. The temperature dependence of the condensate fraction (Fig. 11) and the boson number dependence of the condensation temperature (Fig. 12) shows variations similar to those found for 2d systems. We find that T_0 increases with increasing

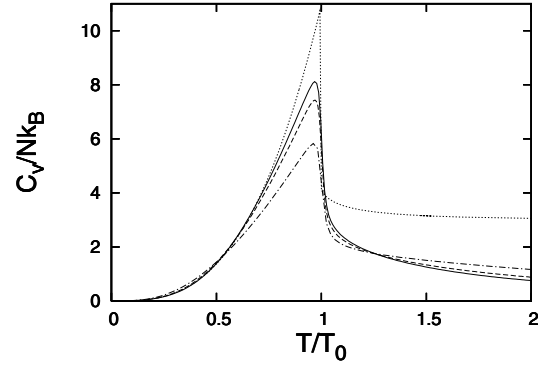


Fig. 13. The effect of harmonic potential strength on the specific heat per boson of 2500 bosons in a $50 \times 50 \times 50$ cubic lattice. The results shown are for $k = 0.01$ (solid line), $k = 0.02$ (dashed line), and $k = 0.05$ (dash-dot line). Dotted line with a sharp peak is for 2500 bosons in a 3d harmonic trap.

harmonic potential strength as well as with the total number of bosons in the lattice. In Figure 13, we have shown the effect of harmonic potential strength on the specific heat of 3d lattice bosons. The peak value in C_v at $T/T_0 \approx 1$ is lower for higher values of k . For the $50 \times 50 \times 50$ lattice, C_v is lower for lower values of k at high temperatures ($T/T_0 > 1.3$). This results from the finite size of the lattice, as mentioned previously. In comparison with a 2d system, the harmonic potential strength has a lesser effect in 3d (see Figs. 9 and 13). In Figure 13, we have also plotted the C_v for free bosons in a harmonic trap. The C_v shows a $(T/T_0)^3$ dependence for $T/T_0 \leq 1$, which is a known result [27]. The figure shows that the C_v for lattice bosons in a harmonic trap has the same temperature dependence in the low temperature range. We have calculated the DOS for the bosons in the 3d lattice (of size $50 \times 50 \times 50$) confined in a harmonic trap and found that it has an E^2 dependence in the low energy range. This accounts for the $(T/T_0)^3$ behavior of the C_v for the lattice bosons in a harmonic trap in the low temperature range.

In the work presented on bosons, the boson-boson interaction is neglected. We believe that our results would approximately hold when interaction energy (U) is much smaller than the hopping energy (t). In an optical lattice, the interaction energy (U) depends on the ratio between the depth of the optical potential to the recoil energy. A weakly interacting regime may be obtained by adjusting the value of the potential depth to a low value [1, 32]. In the weak interaction regime, the interaction induced depletion effects may not be significant.

4 Results on fermions

Due to recent interest [33, 34] in fermions in optical lattices, we have studied single (spin) component Fermi gas in optical lattices with harmonic confinement in 1d, 2d, and 3d. Another interest in studying this fermionic system is that the bosons in the strongly interacting regime behave similar to fermions to avoid double occupancy and

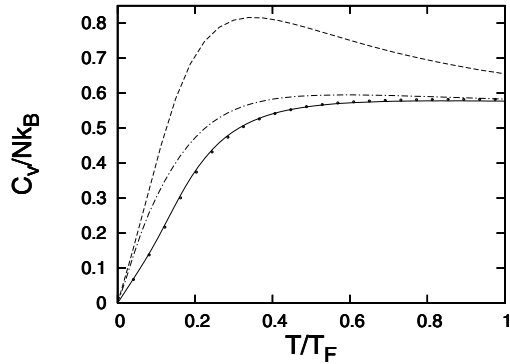


Fig. 14. Specific heat per fermion in a 1d optical lattice (of size 3000 sites) in a harmonic potential with $k = 0.0001$ (dashed line), 0.0005 (dash-dot), 0.01 (solid line). The filled circles are results in the absence of harmonic potential (i.e., for $k = 0$). The number of fermions (N) considered is 150. T_F is the Fermi temperature.

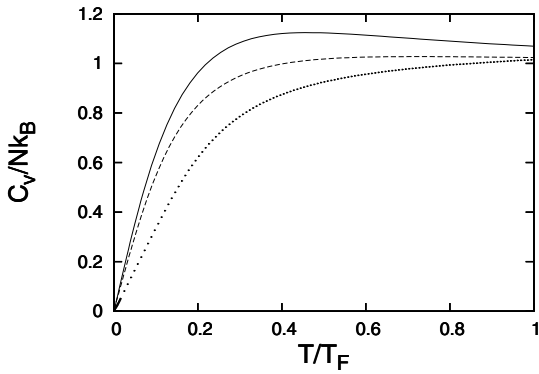


Fig. 15. Specific heat per fermion in a 2d optical lattice (of size 250×250) in a harmonic potential with $k = 0.01$ (solid line), and 0.02 (dashed line). The dots are results in the absence of harmonic potential (i.e., for $k = 0$). The number of fermions considered is 600.

minimize inter-particle repulsion energy and energy spectrum of strongly interacting bosons in 1d is very close to that of the non-interacting fermions [29].

Our results for the specific heat for different values of the trapping strength and for different dimensions are presented in Figures 14–16. The specific heat curves show a linear variation at low temperatures, which is a characteristic of a degenerate Fermi gas. At high temperatures, the specific heat shows a more or less flat behavior. The specific heat increases with decreasing k values except at low temperatures. Similar results are also noted for bosons (see Figs. 4 and 9). With increasing k value, the specific heat for fermions approaches the corresponding value for the $k = 0$ case.

We have determined the gradients of the specific heat curves at low temperatures. For 3d, this value is 9.76 for $k = 0.01$ and 4.98 for $k = 0$. The former is close to the corresponding value (π^2) for free fermions in a harmonic trap while the latter is close to that ($\pi^2/2$) of free fermions. For the 2d case, this slope is 7.2 for $k = 0.01$ and 3.37 for

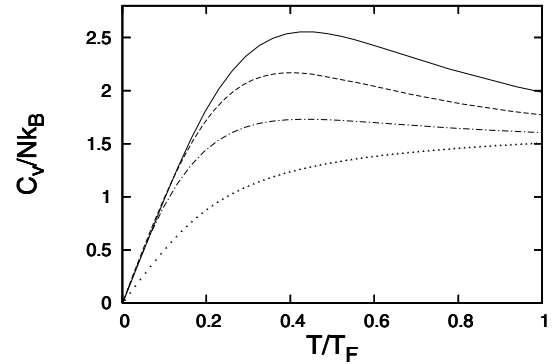


Fig. 16. Specific heat per fermion in a 3d optical lattice (of size $100 \times 100 \times 100$) in a harmonic potential with $k = 0.01$ (solid line), 0.02 (dashed line), and 0.05 (dash-dot line). The dots are results in the absence of harmonic potential (i.e., for $k = 0$). The number of fermions considered is 1000.

$k = 0$, while for 1d it is 3.95 for $k = 0.0001$ and 1.69 for $k = 0$. We find that in all dimensions, the values of the gradient of the specific heat curves at low temperatures for small (k/zt) (where, z is the coordination number of a lattice), are close to those of free fermions in a harmonic trap. The reason behind it lies in the fact that the single particle density of states for small k values in an optical lattice shows the same energy dependence as that of the free fermions (bosons) in a harmonic trap at low energies, as discussed in previous sections.

5 Conclusions

In this paper, we have presented a study of non-interacting bosons under the influence of combined optical lattice and harmonic potentials in one, two, and three dimensions. The condensate fraction in 1d shows a faster reduction with increasing temperature in the low temperature range compared to that in 2d and 3d. The condensation temperature in 1d increases linearly with increasing number of lattice bosons. In comparison, T_0 in 2d and 3d shows a fast growth for small number of bosons and a monotonic increase thereafter. It is found that, for a given number of bosons, the condensation temperature is higher for bosons in the combined harmonic and optical potentials compared to the cases in which either of the potentials is absent.

The specific heat results show several interesting features. In 1d, specific heat of lattice bosons in harmonic potential is found to show a very slow growth with temperature, except in the low temperature range where the growth is relatively faster. In 2d and 3d, C_v of lattice bosons in a harmonic trap has a peak at the condensation temperature. When the lattice size effects are important in C_v , we find that the C_v per boson is lower for smaller lattice sizes except at low temperatures. In 2d, flat regions are observed in C_v for a substantial temperature range below the condensation temperature for relatively smaller lattices. When the lattice size effects are negligible, we find

that the lattice bosons in a harmonic trap has a considerably reduced specific heat compared to that for free bosons in a harmonic trap for temperatures around and above T_0 . In all dimensions, the low temperature C_v is very close to that of free bosons in a harmonic trap. The specific heat of the lattice bosons is strongly dependent on the strength of the harmonic potential in contrast to that of free bosons in a harmonic potential. In all dimensions for large size systems when the lattice size effects are negligible, the C_v is lower for higher values of k around and above T_0 . However, the C_v curves for different k values cross each other above T_0 when finite size effects are present. Considering recent interest in fermions in optical lattices, we presented some preliminary results on the specific heat of fermions in optical lattices in a harmonic potential in 1d, 2d, and 3d. The specific heat is found to show a linear temperature dependence at low temperatures and a flat behavior at high temperatures. With increasing strength of the harmonic potential, the specific heat is found to approach the pure lattice fermion results. The temperature dependence of the specific heat of lattice bosons and fermions in a harmonic trap is governed by a complex interplay of the delocalizing effects of the boson kinetic energy, confining effect of the harmonic potential, and the thermal energy.

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